# ON THE THEORY OF BENDING OF ANISOTROPIC PLATES AND SHALLOW SHELLS

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*PMM Vol.24, No.2, 1960, pp. 350-360*

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*(Received 3 July 1959)*

**1.** Let us consider a thin orthotropic shell of constant thickness $h$. Let the material of the shell obey the generalized Hooke's law and let it have at each point three planes of elastic symmetry whose principal directions coincide with the directions of the coordinate lines $a$, $\beta$, $\gamma$. The middle surface of the shell is taken as a coordinate surface and it is referred to curvilinear orthogonal coordinates $a$ and $\beta$ coinciding with lines of principal curvature of the middle surface. The third coordinate line $\gamma$ is rectilinear and represents the distance along the normal from the point $(a, \beta)$ of the middle surface to the point $(a, \beta, \gamma)$ of the shell.

Renouncing the hypothesis on undeformable normals, we introduce the following assumptions:

a) The distance along the normal $(\gamma)$ between two points of the shell after deformation remains unchanged;

b) The shear stresses $\tau_{\alpha\gamma}$ and $\tau_{\beta\gamma}$ vary in accordance with a specified law through the thickness of the shell.

**2.** For greater clarity we first present the suggested method for a plate ($k_1 = 0$, $k_2 = 0$, $a$, $\beta$, $\gamma$ are rectilinear orthogonal coordinates).

The basic assumptions will be written down in the following form:

a) we assume approximately

$$e_\gamma = 0 \tag{2.1}$$

b) the shear stresses $\tau_{\alpha\gamma}$ and $\tau_{\beta\gamma}$ are of the form

$$\tau_{\alpha\gamma} = f_1(\gamma)\varphi(\alpha,\beta) + \frac{\gamma}{h}(X^+ + X^-) + \frac{X^+ - X^-}{2}$$
$$\tau_{\beta\gamma} = f_2(\gamma)\psi(\alpha,\beta) + \frac{\gamma}{h}(Y^+ + Y^-) + \frac{Y^+ - Y^-}{2} \tag{2.2}$$

where $X^+$, ..., $Y^-$ are the components along the axes of a moving trihedron (along the directions of positive tangents to the lines $\beta$ = const, $a$ = const) of the vectors of intensity of surface loads, applied upon the outer faces of the plate $\gamma = 1/2\ h$ and $\gamma = -\ 1/2\ h$, $\phi(a, \beta)$, $\psi(a, \beta)$ are arbitrary functions of coordinates $a$, $\beta$ to be found; $f_i(y)$ are functions, characterizing the variation laws of shear stresses $\tau_{\alpha\gamma}$ and $\tau_{\beta\gamma}$ through the thickness, whereby $f_i(\pm\ 1/2\ h) = 0$.

From the equations of the generalized Hooke's law we have

$$\sigma_\alpha = B_{11}e_\alpha + B_{12}e_\beta - A_1\sigma_\gamma, \quad \sigma_\beta = B_{22}e_\beta + B_{12}e_\alpha - A_2\sigma_\gamma \tag{2.3}$$

$$\tau_{\alpha\gamma} = B_{55}e_{\alpha\gamma}, \quad \tau_{\beta\gamma} = B_{44}e_{\beta\gamma}, \quad \tau_{\alpha\beta} = B_{66}e_{\alpha\beta} \tag{2.4}$$

where*

$$B_{11} = \frac{E_1}{1-\nu_1\nu_2}, \quad B_{22} = \frac{E_2}{1-\nu_1\nu_2}, \quad B_{66} = G_{12}, \quad B_{55} = G_{13}, \quad B_{44} = G_{23}$$

$$A_1 = -\frac{E_1}{E_3}\frac{\nu_{13}+\nu_2\nu_{23}}{1-\nu_1\nu_2}, \quad A_2 = -\frac{E_2}{E_3}\frac{\nu_{23}+\nu_1\nu_{13}}{1-\nu_1\nu_2}, \quad B_{12} = \frac{\nu_2E_1}{1-\nu_1\nu_2} = \frac{\nu_1E_2}{1-\nu_1\nu_2} \tag{2.5}$$

From the equations of the theory of elasticity we have for the components of strain

$$e_\alpha = \frac{\partial u_\alpha}{\partial\alpha}, \quad e_\beta = \frac{\partial u_\beta}{\partial\beta}, \quad e_{\alpha\beta} = \frac{\partial u_\alpha}{\partial\beta} + \frac{\partial u_\beta}{\partial\alpha} \tag{2.6}$$

$$e_\gamma = \frac{\partial u_\gamma}{\partial\gamma}, \quad e_{\alpha\gamma} = \frac{\partial u_\alpha}{\partial\gamma} + \frac{\partial u_\gamma}{\partial\alpha}, \quad e_{\beta\gamma} = \frac{\partial u_\beta}{\partial\gamma} + \frac{\partial u_\gamma}{\partial\beta} \tag{2.7}$$

From relations (2.7), by virtue of (2.1), (2.2) and (2.4) we obtain for the displacements of the plate

$$u_\alpha = u(\alpha,\beta) - \gamma\frac{\partial w}{\partial\alpha} + \gamma X_1 + \frac{\gamma^2}{2h}X_2 + J_{01}(\gamma)\Phi_1 \tag{2.8}$$
$$u_\beta = v(\alpha,\beta) - \gamma\frac{\partial w}{\partial\beta} + \gamma Y_1 + \frac{\gamma^2}{2h}Y_2 + J_{02}(\gamma)\Phi_2$$

$$u_\gamma = w(\alpha,\beta) \tag{2.9}$$

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* Here and in the following the usual notation is used for the elastic constants [1,2].

where

$$X_1 = \frac{1}{G_{13}}\frac{X^+ - X^-}{2}, \qquad Y_1 = \frac{1}{G_{23}}\frac{Y^+ - Y^-}{2}, \qquad \Phi_1 = \frac{1}{G_{13}}\varphi(\alpha, \beta)$$
$$X_2 = \frac{1}{C_{13}}(X^+ - X^-), \qquad Y_2 = \frac{1}{G_{23}}(Y^+ - Y^-), \qquad \Phi_2 = \frac{1}{G_{23}}\psi(\alpha, \beta) \tag{2.10}$$

$$J_{01} = \int_0^\gamma f_1(\gamma)\, d\gamma, \qquad J_{02} = \int_0^\gamma f_2(\gamma)\, d\gamma \tag{2.11}$$

$u(\alpha, \beta)$, $v(\alpha, \beta)$, $w(\alpha, \beta)$ are the tangential and normal displacements of the middle surface of the shell.

From the third equation of equilibrium of a differential element of the plate [2] we obtain

$$\sigma_\gamma = \frac{B_{55}}{2}\left[J_{01}\left(\frac{h}{2}\right) + J_{01}\left(-\frac{h}{2}\right)\right]\frac{\partial\Phi_1}{\partial\alpha} + \frac{B_{44}}{2}\left[J_{02}\left(\frac{h}{2}\right) + J_{02}\left(-\frac{h}{2}\right)\right]\frac{\partial\Phi_2}{\partial\beta} +$$
$$+ \frac{h}{8}\left(B_{55}\frac{\partial X_2}{\partial\alpha} + B_{44}\frac{\partial Y_2}{\partial\beta}\right) - B_{55}\left[J_{01}(\gamma)\frac{\partial\Phi_1}{\partial\alpha} + \gamma\frac{\partial X_1}{\partial\alpha} + \frac{\gamma^2}{2h}\frac{\partial X_2}{\partial\alpha}\right] -$$
$$- B_{44}\left[J_{02}(\gamma)\frac{\partial\Phi_2}{\partial\beta} + \gamma\frac{\partial Y_1}{\partial\beta} + \frac{\gamma^2}{2h}\frac{\partial Y_2}{\partial\beta}\right] + Z_1 \tag{2.12}$$

$$B_{55}\left[J_{01}\left(\frac{h}{2}\right) - J_{01}\left(-\frac{h}{2}\right)\right]\frac{\partial\Phi_1}{\partial\alpha} + B_{44}\left[J_{02}\left(\frac{h}{2}\right) - J_{02}\left(-\frac{h}{2}\right)\right]\frac{\partial\Phi_2}{\partial\beta} =$$
$$= -h\left(B_{55}\frac{\partial X_1}{\partial\alpha} + B_{44}\frac{\partial Y_1}{\partial\beta}\right) - Z_2 \tag{2.13}$$

where

$$Z_1 = \frac{Z^+ - Z^-}{2}, \qquad Z_2 = Z^+ + Z^- \tag{2.14}$$

$Z^+$, $Z^-$ are the normal components of the vectors of intensity of surface loads, applied at the external faces of the plate ($\gamma = 1/2\, h$, $\gamma = -1/2\, h$).

Equation (2.13) is the third integral equation of equilibrium. From (2.3) and (2.4), by virtue of (2.6), (2.8), (2.9) and (2.12), we obtain expressions for $\sigma_\alpha$, $\sigma_\beta$ and $\tau_{\alpha\beta}$, which will not be presented here.

Substituting the expressions for the stresses $\sigma_\alpha$, $\sigma_\beta$, $\tau_{\alpha\beta}$, $\tau_{\alpha\gamma}$ and $\tau_{\beta\gamma}$ into the usual formulas for the internal forces and moments we obtain

$$T_1 = B_{11}\left(h\frac{\partial u}{\partial\alpha} + \frac{h^2}{24}\frac{\partial X_2}{\partial\alpha} + J_1\frac{\partial\Phi_1}{\partial\alpha}\right) + B_{12}\left(h\frac{\partial v}{\partial\beta} + \frac{h^2}{24}\frac{\partial Y_2}{\partial\beta} + J_2\frac{\partial\Phi_2}{\partial\beta}\right) +$$
$$+ A_1\left[B_{55}\left(J_1\frac{\partial\Phi_1}{\partial\alpha} + \frac{h^2}{24}\frac{\partial X_2}{\partial\alpha}\right) + B_{44}\left(J_2\frac{\partial\Phi_2}{\partial\beta} + \frac{h^2}{24}\frac{\partial Y_2}{\partial\beta}\right) - h\chi(\alpha, \beta)\right] \tag{2.15}$$

$$T_2 = B_{22}\left(h\frac{\partial v}{\partial \beta} + \frac{h^2}{24}\frac{\partial X_2}{\partial \beta} + J_2\frac{\partial \Phi_2}{\partial \beta}\right) + B_{12}\left(h\frac{\partial u}{\partial \alpha} + \frac{h^2}{24}\frac{\partial X_2}{\partial \alpha} + J_1\frac{\partial \Phi_1}{\partial \alpha}\right) +$$

$$+ A_2\left[B_{44}\left(J_2\frac{\partial \Phi_2}{\partial \beta} + \frac{h^2}{24}\frac{\partial Y_2}{\partial \beta}\right) + B_{55}\left(J_1\frac{\partial \Phi_1}{\partial \alpha} + \frac{h^2}{24}\frac{\partial X_2}{\partial \alpha}\right) - h\chi(\alpha, \beta)\right] \quad (2.16)$$

$$S = B_{66}\left[h\left(\frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \alpha}\right) + \frac{h^2}{24}\left(\frac{\partial X_2}{\partial \beta} + \frac{\partial Y_2}{\partial \alpha}\right) + J_1\frac{\partial \Phi_1}{\partial \beta^2} + J_2\frac{\partial \Phi_2}{\partial \alpha}\right] \quad (2.17)$$

$$M_1 = B_{11}\left(-\frac{h^3}{12}\frac{\partial^2 w}{\partial \alpha^2} + \frac{h^3}{12}\frac{\partial X_1}{\partial \alpha} + J_3\frac{\partial \Phi_1}{\partial \alpha}\right) + B_{12}\left(-\frac{h^3}{12}\frac{\partial^2 w}{\partial \beta^2} + \frac{h^3}{12}\frac{\partial Y_1}{\partial \beta} +\right.$$

$$\left. + J_4\frac{\partial \Phi_2}{\partial \beta}\right) + A_1\left[B_{55}\left(J_3\frac{\partial \Phi_1}{\partial \alpha} + \frac{h^3}{12}\frac{\partial X_1}{\partial \alpha}\right) + B_{44}\left(J_4\frac{\partial \Phi_2}{\partial \beta} + \frac{h^3}{12}\frac{\partial Y_1}{\partial \beta}\right)\right] \quad (2.18)$$

$$M_2 = B_{22}\left(-\frac{h^3}{12}\frac{\partial^2 w}{\partial \beta^2} + \frac{h^3}{12}\frac{\partial Y_1}{\partial \beta} + J_4\frac{\partial \Phi_2}{\partial \beta}\right) + B_{12}\left(-\frac{h^3}{12}\frac{\partial^2 w}{\partial \alpha^2} + \frac{h^3}{12}\frac{\partial X_1}{\partial \alpha} + J_3\frac{\partial \Phi_1}{\partial \alpha}\right) +$$

$$+ A_2\left[B_{44}\left(J_4\frac{\partial \Phi_2}{\partial \beta} + \frac{h^3}{12}\frac{\partial Y_1}{\partial \beta}\right) + B_{55}\left(J_3\frac{\partial \Phi_1}{\partial \alpha} + \frac{h^3}{12}\frac{\partial X_1}{\partial \alpha}\right)\right] \quad (2.19)$$

$$H = B_{66}\left[-2\frac{h^3}{12}\frac{\partial^2 w}{\partial \alpha\,\partial \beta} + \frac{h^3}{12}\left(\frac{\partial X_1}{\partial \beta} + \frac{\partial Y_1}{\partial \alpha}\right) + J_3\frac{\partial \Phi_1}{\partial \beta} + J_4\frac{\partial \Phi_2}{\partial \alpha}\right] \quad (2.20)$$

$$N_1 = B_{55}(J_5\Phi_1 + hX_1), \qquad N_2 = B_{44}(J_6\Phi_2 + hY_1) \quad (2.21)$$

where the following notations are introduced:

$$\chi(\alpha, \beta) = \frac{B_{55}}{2}\left[J_{01}(^1/_2 h) + J_{01}(-^1/_2 h)\right]\frac{\partial \Phi_1}{\partial \alpha} + \frac{B_{44}}{2}\left[J_{02}(^1/_2 h) + J_{02}(-^1/_2 h)\right]\frac{\partial \Phi_2}{\partial \beta} +$$

$$+ \frac{h}{8}\left(B_{55}\frac{\partial X_2}{\partial \alpha} + B_{44}\frac{\partial Y_2}{\partial \beta}\right) + Z_1 \quad (2.22)$$

$$J_1 = \int_{-^1/_2 h}^{^1/_2 h} J_{01}(\gamma)\,d\gamma, \quad J_3 = \int_{-^1/_2 h}^{^1/_2 h} \gamma J_{01}(\gamma)\,d\gamma, \quad J_5 = \int_{-^1/_2 h}^{^1/_2 h} f_1(\gamma)\,d\gamma$$

$$J_2 = \int_{-^1/_2 h}^{^1/_2 h} J_{02}(\gamma)\,d\gamma, \quad J_4 = \int_{-^1/_2 h}^{^1/_2 h} \gamma J_{02}(\gamma)\,d\gamma, \quad J_6 = \int_{-^1/_2 h}^{^1/_2 h} f_2(\gamma)\,d\gamma \quad (2.23)$$

The equations of equilibrium of an element of the shell are of the form

$$\frac{\partial T_1}{\partial \alpha} + \frac{\partial S}{\partial \beta} = -B_{55}X_2, \qquad \frac{\partial T_2}{\partial \beta} + \frac{\partial S}{\partial \alpha} = -B_{44}Y_2 \quad (2.24)$$

$$\frac{\partial M_1}{\partial \alpha} + \frac{\partial H}{\partial \beta} = N_1, \qquad \frac{\partial M_2}{\partial \beta} + \frac{\partial H}{\partial \alpha} = N_2, \qquad \frac{\partial N_1}{\partial \alpha} + \frac{\partial N_2}{\partial \beta} = -Z_2$$

Substituting the values of the internal forces and moments from

(2.15)-(2.21) into Equations (2.24) and taking into account (2.10) and (2.11), we obtain

$$L_{11}(C_{ik})u+L_{12}(C_{ik})v+a_{55}J_1L_{11}(B_{ik})\varphi+a_{44}J_2L_{12}(B_{ik})\psi-$$
$$-A_1\left[\left\{\frac{h}{2}\left[J_{01}\left(\frac{h}{2}\right)+J_{01}\left(-\frac{h}{2}\right)\right]-J_1\right\}\frac{\partial^2\varphi}{\partial\alpha^2}+\left\{\frac{h}{2}\left[J_{02}\left(\frac{h}{2}\right)+J_{02}\left(-\frac{h}{2}\right)\right]-\right.\right.$$
$$\left.\left.-J_2\right\}\frac{\partial^2\psi}{\partial\alpha\partial\beta}\right]=-\frac{h^2}{24}[L_{11}(B_{ik})X_2+L_{12}(B_{ik})Y_2]-$$
$$-B_{55}X_2+A_1\left[\frac{h^2}{12}\left(B_{55}\frac{\partial^2X_2}{\partial\alpha^2}+B_{44}\frac{\partial^2Y_2}{\partial\alpha\partial\beta}\right)+h\frac{\partial Z_1}{\partial\alpha}\right] \tag{2.25}$$

$$L_{22}(C_{ik})v+L_{12}(C_{ik})u+a_{44}J_2L_{22}(B_{ik})\psi+a_{55}J_1L_{12}(B_{ik})\varphi-$$
$$-A_2\left[\left\{\frac{h}{2}\left[J_{02}\left(\frac{h}{2}\right)+J_{02}\left(-\frac{h}{2}\right)\right]-J_2\right\}\frac{\partial^2\psi}{\partial\beta^2}+\left\{\frac{h}{2}\left[J_{01}\left(\frac{h}{2}\right)+J_{01}\left(-\frac{h}{2}\right)\right]-\right.\right.$$
$$\left.\left.-J_1\right\}\frac{\partial^2\varphi}{\partial\alpha\partial\beta}\right]=-\frac{h^2}{24}[L_{22}(B_{ik})Y_2+L_{12}(B_{ik})X_2]-B_{44}Y_2+A_2\left[\frac{h^2}{12}\left(B_{44}\frac{\partial^2Y_2}{\partial\beta^2}+\right.\right.$$
$$\left.\left.+B_{55}\frac{\partial^2X_2}{\partial\alpha\partial\beta}\right)+h\frac{\partial Z_1}{\partial\beta}\right] \tag{2.26}$$

$$\left[J_{01}\left(\frac{h}{2}\right)-J_{01}\left(-\frac{h}{2}\right)\right]\frac{\partial\varphi}{\partial\alpha}+\left[J_{02}\left(\frac{h}{2}\right)-J_{02}\left(-\frac{h}{2}\right)\right]\frac{\partial\psi}{\partial\beta}=$$
$$=-Z_2-h\left(B_{55}\frac{\partial X_1}{\partial\alpha}+B_{44}\frac{\partial Y_1}{\partial\beta}\right) \tag{2.27}$$

$$L_{13}(D_{ik})w-a_{55}J_3L_{11}(B_{ik})\varphi-a_{44}J_4L_{12}(B_{ik})\psi+\left[J_{01}\left(\frac{h}{2}\right)-J_{01}\left(-\frac{h}{2}\right)\right]\varphi-$$
$$-A_1\left(J_3\frac{\partial^2\varphi}{\partial\alpha^2}+J_4\frac{\partial^2\psi}{\partial\alpha\partial\beta}\right)=\frac{h^3}{12}[L_{11}(B_{ik})X_1+L_{12}(B_{ik})Y_1]-hB_{55}X_1+$$
$$+A_1\frac{h^3}{12}\left(B_{55}\frac{\partial^2X_1}{\partial\alpha^2}+B_{44}\frac{\partial^2Y_1}{\partial\alpha\partial\beta}\right) \tag{2.28}$$

$$L_{23}(D_{ik})w-a_{44}J_4L_{22}(B_{ik})\psi-a_{55}J_3L_{12}(B_{ik})\varphi+\left[J_{02}\left(\frac{h}{2}\right)-J_{02}\left(-\frac{h}{2}\right)\right]\psi-$$
$$-A_2\left(J_3\frac{\partial^2\psi}{\partial\beta^2}+J_4\frac{\partial^2\varphi}{\partial\alpha\partial\beta}\right)=\frac{h^3}{12}[L_{22}(B_{ik})Y_1+L_{12}(B_{ik})X_1]-$$
$$-hB_{44}Y_1+A_2\frac{h^3}{12}\left(B_{44}\frac{\partial^2Y_1}{\partial\beta^2}+B_{55}\frac{\partial^2X_1}{\partial\alpha\partial\beta}\right) \tag{2.29}$$

where

$$L_{11}(a_{ik})=a_{11}\frac{\partial^2}{\partial\alpha^2}+a_{66}\frac{\partial^2}{\partial\beta^2},\ L_{22}(a_{ik})=a_{22}\frac{\partial^2}{\partial\beta^2}+a_{66}\frac{\partial^2}{\partial\alpha^2} \tag{2.30}$$

$$L_{12}(a_{ik})=(a_{12}+a_{66})\frac{\partial^2}{\partial\alpha\partial\beta},\qquad D_{ik}=B_{ik}\frac{h^3}{12},\quad C_{ik}=B_{ik}h$$

$$L_{13}(a_{ik})=a_{11}\frac{\partial^3}{\partial\alpha^3}+(a_{12}+2a_{66})\frac{\partial^3}{\partial\alpha\partial\beta^2},\quad L_{23}(a_{ik})=a_{22}\frac{\partial^3}{\partial\beta^3}+(a_{12}+2a_{66})\frac{\partial^3}{\partial\beta\partial\alpha^2}$$

Equations (2.25) through (2.29) constitute a complete system of five differential equations with respect to five unknown functions $u$, $v$, $w$, $\phi$, $\psi$, by means of which all relevant quantities of the plate may be expressed.

The boundary conditions may be represented in the usual form [3,4].

**3.** The case when $X^{\pm} = 0$ and $Y^{\pm} = 0$ is of considerable practical interest, that is, the case when the plate is subjected only to normal loading $Z^{\pm}$. In this case assuming also that [3]

$$f_i(\gamma) = \frac{1}{2}\left(\frac{1}{4}h^2 - \gamma^2\right) \tag{3.1}$$

Equations (2.25)-(2.29) take on the form

$$L_{11}(C_{ik})\,u + L_{12}(C_{ik})\,v = hA_1\frac{\partial Z_1}{\partial\alpha}, \quad L_{22}(C_{ik})\,v + L_{12}(C_{ik})\,u = hA_2\frac{\partial Z_1}{\partial\beta} \tag{3.2}$$

$$\frac{\partial\varphi}{\partial\alpha} + \frac{\partial\psi}{\partial\beta} = -\frac{12}{h^3}Z_2 \tag{3.3}$$

$$L_{13}(D_{ik})\,w - \frac{h^2}{10}\left[a_{55}L_{11}(D_{ik})\,\varphi + a_{44}L_{12}(D_{ik})\,\psi\right] + \frac{h^3}{12}\varphi = -A_1\frac{h^2}{10}\frac{\partial Z_2}{\partial\alpha} \tag{3.4}$$

$$L_{23}(D_{ik})\,w - \frac{h^2}{10}\left[a_{44}L_{22}(D_{ik})\,\psi + a_{55}L_{12}(D_{ik})\,\varphi\right] + \frac{h^3}{12}\psi = -A_2\frac{h^2}{10}\frac{\partial Z_2}{\partial\beta} \tag{3.5}$$

Then the relevant quantities are

$$T_1 = C_{11}\frac{\partial u}{\partial\alpha} + C_{12}\frac{\partial v}{\partial\beta} - hA_1Z_1, \qquad S = C_{66}\left(\frac{\partial u}{\partial\beta} + \frac{\partial v}{\partial\alpha}\right)$$
$$T_2 = C_{22}\frac{\partial v}{\partial\beta} + C_{12}\frac{\partial u}{\partial\alpha} - hA_2Z_1, \tag{3.6}$$

$$M_1 = -D_{11}\frac{\partial^2 w}{\partial\alpha^2} - D_{12}\frac{\partial^2 w}{\partial\beta^2} + \frac{h^2}{10}\left(\frac{D_{11}}{B_{55}}\frac{\partial\varphi}{\partial\alpha} + \frac{D_{12}}{B_{44}}\frac{\partial\psi}{\partial\beta}\right) - \frac{h^2}{10}A_1Z_2$$

$$M_2 = -D_{22}\frac{\partial^2 w}{\partial\beta^2} - D_{12}\frac{\partial^2 w}{\partial\alpha^2} + \frac{h^2}{10}\left(\frac{D_{22}}{B_{44}}\frac{\partial\psi}{\partial\beta} + \frac{D_{12}}{B_{55}}\frac{\partial\varphi}{\partial\alpha}\right) - \frac{h^2}{10}A_2Z_2 \tag{3.7}$$

$$H = D_{66}\left[-2\frac{\partial^2 w}{\partial\alpha\partial\beta} + \frac{h^2}{10}\left(\frac{1}{B_{55}}\frac{\partial\varphi}{\partial\beta} + \frac{1}{B_{44}}\frac{\partial\psi}{\partial\alpha}\right)\right], \quad N_1 = \frac{h^3}{12}\varphi, \quad N_2 = \frac{h^3}{12}\psi$$

$$\sigma_\alpha = B_{11}\left(\frac{\partial u}{\partial\alpha} - \gamma\frac{\partial^2 w}{\partial\alpha^2}\right) + B_{12}\left(\frac{\partial v}{\partial\beta} - \gamma\frac{\partial^2 w}{\partial\beta^2}\right) + \frac{1}{2}\left(\gamma\frac{h^2}{4} - \frac{\gamma^3}{3}\right)\left(\frac{B_{11}}{B_{55}}\frac{\partial\varphi}{\partial\alpha} + \frac{B_{12}}{B_{44}}\frac{\partial\psi}{\partial\beta}\right) -$$
$$- A_1\left[Z_1 + 6\left(\frac{1}{4}\frac{\gamma}{h} - \frac{1}{3}\frac{\gamma^3}{h^3}\right)Z_2\right] \tag{3.8}$$

$$\sigma_\beta = B_{22}\left(\frac{\partial v}{\partial\beta} - \gamma\frac{\partial^2 w}{\partial\beta^2}\right) + B_{12}\left(\frac{\partial u}{\partial\alpha} - \gamma\frac{\partial^2 w}{\partial\alpha^2}\right) + \frac{1}{2}\left(\gamma\frac{h^2}{4} - \frac{\gamma^3}{3}\right)\left(\frac{B_{22}}{B_{44}}\frac{\partial\psi}{\partial\beta} + \frac{B_{12}}{B_{55}}\frac{\partial\varphi}{\partial\alpha}\right) -$$
$$- A_2\left[Z_1 + 6\left(\frac{1}{4}\frac{\gamma}{h} - \frac{1}{3}\frac{\gamma^3}{h^3}\right)Z_2\right] \tag{3.9}$$

$$\tau_{\alpha\beta} = B_{66}\left[\frac{\partial u}{\partial\beta} + \frac{\partial v}{\partial\alpha} - 2\gamma\frac{\partial^2 w}{\partial\alpha\partial\beta} + \frac{1}{2}\left(\gamma\frac{h^2}{4} - \frac{\gamma^3}{3}\right)\left(\frac{1}{B_{55}}\frac{\partial\varphi}{\partial\beta} + \frac{1}{B_{44}}\frac{\partial\psi}{\partial\alpha}\right)\right] \tag{3.10}$$

**4.** In the case of a shallow orthotropic shell we assume approximately that the internal geometry of the middle surface of the shell of non-vanishing Gaussian curvature does not differ from the Euclidian geometry on the plane, that is, for a suitably chosen absolute system of coordinates the coefficients of the first quadratic form are [2,5]

$$A = 1, \qquad B = 1 \tag{4.1}$$

With the same degree of accuracy we assume that the principal curvatures of the middle surface behave as constant quantities

$$k_1 = \text{const}, \qquad k_2 = \text{const} \tag{4.2}$$

The theory of very shallow shells taking into account the effect of transverse shear and normal stresses $\sigma_\gamma$, will be constructed on the basis of assumptions introduced in Section 1, which will be written down in the following form:

a) we assume approximately

$$e_\gamma = 0 \tag{4.3}$$

b) the shear stresses are of the form [3,4]

$$\tau_{\alpha\gamma} = \frac{1}{2}\left(\frac{1}{4}h^2 - \gamma^2\right)\varphi(\alpha, \beta), \qquad \tau_{\beta\gamma} = \frac{1}{2}\left(\frac{1}{4}h^2 - \gamma^2\right)\psi(\alpha, \beta) \tag{4.4}$$

where, as before, $\phi = \phi(\alpha, \beta)$ and $\psi = \psi(\alpha, \beta)$ are arbitrary unknown functions of the coordinates $\alpha\beta$.

For the sake of simplicity it is assumed here that the shell is subjected only to a normal loading $Z = Z^+$ and that the shear stresses $\tau_{\alpha\gamma}$ and $\tau_{\beta\gamma}$ vary through the thickness of the shell, following the parabolic law (3.1).

From the theory of elasticity the components of strain are

$$e_\alpha = \frac{\partial u_\alpha}{\partial \alpha} + k_1 u_\gamma, \qquad e_\beta = \frac{\partial u_\beta}{\partial \beta} + k_2 u_\gamma, \qquad e_{\alpha\beta} = \frac{\partial u_\alpha}{\partial \beta} + \frac{\partial u_\beta}{\partial \alpha}$$
$$e_\gamma = \frac{\partial u_\gamma}{\partial \gamma}, \qquad e_{\alpha\gamma} = \frac{\partial u_\alpha}{\partial \gamma} + \frac{\partial u_\gamma}{\partial \alpha}, \qquad e_{\beta\gamma} = \frac{\partial u_\beta}{\partial \gamma} + \frac{\partial u_\gamma}{\partial \beta} \tag{4.5}$$

Here, and in the following, quantities of the order $hk_i$ will be neglected as compared to unity on the basis of pronounced shallowness of the thin shell considered; this will be done whereever it is obvious. The shell is assumed to be thin but the thickness $h$ is still a finite quantity.

Taking (4.4) into account, the generalized Hooke's law, which in the case of the shell considered is of the form (2.3) and (2.4), the displacements of any point of the shell in accordance with (4.5) will be

$$u_\alpha = u - \gamma\frac{\partial w}{\partial \alpha} + \frac{\gamma}{2B_{55}}\left(\frac{h^2}{4} - \frac{\gamma^2}{3}\right)\varphi, \; u_\beta = v - \gamma\frac{\partial w}{\partial \beta} + \frac{\gamma}{2B_{44}}\left(\frac{h^2}{4} - \frac{\gamma^2}{3}\right)\psi, \; u_\gamma = w \tag{4.6}$$

where $u = u(\alpha, \beta)$, $v = v(\alpha, \beta)$, $w = w(\alpha, \beta)$ are the tangential and normal displacements of the corresponding point of the middle surface of the shell. From (4.6), substituting the values of $u_\alpha$, $u_\beta$, $u_\gamma$ into (4.5) and further into (2.3) and (2.4), we obtain the following expressions for the stresses

$$\sigma_\alpha = B_{11}\frac{\partial u}{\partial \alpha} + B_{12}\frac{\partial v}{\partial \beta} - \gamma\left(B_{11}\frac{\partial^2 w}{\partial \alpha^2} + B_{12}\frac{\partial^2 w}{\partial \beta^2}\right) + (k_1 B_{11} + k_2 B_{12})\, w +$$
$$+ \frac{\gamma}{2}\left(\frac{h^2}{4} - \frac{\gamma^2}{3}\right)\left(B_{11}a_{55}\frac{\partial \varphi}{\partial \alpha} + B_{12}a_{44}\frac{\partial \psi}{\partial \beta}\right) - A_1\sigma_\gamma \tag{4.7}$$

$$\sigma_\beta = B_{22}\frac{\partial v}{\partial \beta} + B_{12}\frac{\partial u}{\partial \alpha} - \gamma\left(B_{22}\frac{\partial^2 w}{\partial \beta^2} + B_{12}\frac{\partial^2 w}{\partial \alpha^2}\right) + (k_2 B_{22} + k_1 B_{12})\, w +$$
$$+ \frac{\gamma}{2}\left(\frac{h^2}{4} - \frac{\gamma^2}{3}\right)\left(B_{22}a_{44}\frac{\partial \psi}{\partial \beta} + B_{12}a_{55}\frac{\partial \varphi}{\partial \alpha}\right) - A_2\sigma_\gamma \tag{4.8}$$

$$\tau_{\alpha\beta} = B_{66}\left(\frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \alpha}\right) - 2B_{66}\gamma\frac{\partial^2 w}{\partial \alpha \partial \beta} + \frac{\gamma}{2}\left(\frac{h^2}{4} - \frac{\gamma^2}{3}\right)\left(B_{66}a_{55}\frac{\partial \varphi}{\partial \beta} + B_{66}a_{44}\frac{\partial \psi}{\partial \alpha}\right) \tag{4.9}$$

Substituting the relations for the stresses $\sigma_\alpha$, $\sigma_\beta$, $\tau_{\alpha\gamma}$ and $\tau_{\beta\gamma}$ into the third differential equation of equilibrium [2], expressed in terms of curvilinear coordinates, and integrating with respect to $\gamma$, taking into account that $\sigma_\gamma = Z = Z^+$ for $\gamma = 1/2\, h$ and $\sigma_\gamma = 0$ for $\gamma = -1/2 h$, with the accuracy within the theory of very shallow shells, we obtain for the stress $\sigma_\gamma$

$$\sigma_\gamma = \frac{Z}{2} + \gamma\left[(k_1 B_{11} + k_2 B_{12})\frac{\partial u}{\partial \alpha} + (k_2 B_{22} + k_1 B_{12})\frac{\partial v}{\partial \beta} +\right.$$
$$\left. + (k_1^2 B_{11} + 2k_1 k_2 B_{12} + k_2^2 B_{22})\, w\right] + \frac{1}{2}\left(\frac{h^2}{4} - \gamma^2\right)\left[(k_1 B_{11} + k_2 B_{12})\frac{\partial^2 w}{\partial \alpha^2} +\right.$$
$$\left. + (k_2 B_{22} + k_1 B_{12})\frac{\partial^2 w}{\partial \beta^2}\right] - \frac{\gamma}{2}\left(\frac{h^2}{4} - \frac{\gamma^2}{3}\right)\left(\frac{\partial \varphi}{\partial \alpha} + \frac{\partial \psi}{\partial \beta}\right) \tag{4.10}$$

Substituting the expressions for the stresses $\sigma_\alpha$, $\sigma_\beta$, $\tau_{\alpha\beta}$, $\tau_{\alpha\gamma}$, $\tau_{\beta\gamma}$ into the usual formulas for the internal forces and moments we obtain

$$T_1 = C_{11}\left(\frac{\partial u}{\partial \alpha} + k_1 w\right) + C_{12}\left(\frac{\partial v}{\partial \beta} + k_2 w\right) - A_1 T^* \tag{4.11}$$

$$T_2 = C_{22}\left(\frac{\partial v}{\partial \beta} + k_2 w\right) + C_{12}\left(\frac{\partial u}{\partial \alpha} + k_1 w\right) - A_2 T^* \tag{4.12}$$

$$S = C_{66}\left(\frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \alpha}\right) \tag{4.13}$$

$$M_1 = -D_{11}\frac{\partial^2 w}{\partial \alpha^2} - D_{12}\frac{\partial^2 w}{\partial \beta^2} + \frac{h^2}{10}\left(a_{55}D_{11}\frac{\partial \varphi}{\partial \alpha} + a_{44}D_{12}\frac{\partial \psi}{\partial \beta}\right) - A_1 M^* \tag{4.14}$$

$$M_2 = -D_{22}\frac{\partial^2 w}{\partial \beta^2} - D_{12}\frac{\partial^2 w}{\partial \alpha^2} + \frac{h^2}{10}\left(a_{44}D_{22}\frac{\partial \psi}{\partial \beta} + a_{55}D_{12}\frac{\partial \varphi}{\partial \alpha}\right) - A_2 M^* \tag{4.15}$$

$$H = -2D_{66}\frac{\partial^2 w}{\partial \alpha \partial \beta} + \frac{h^2}{10}D_{66}\left(a_{55}\frac{\partial \varphi}{\partial \beta} + a_{44}\frac{\partial \psi}{\partial \alpha}\right), \quad N_1 = \frac{h^3}{12}\varphi, \quad N_2 = \frac{h^3}{12}\psi \tag{4.16}$$

where

$$T^* = \frac{h}{2}Z + (k_1 D_{11} + k_2 D_{12})\frac{\partial^2 w}{\partial \alpha^2} + (k_2 D_{22} + k_1 D_{12})\frac{\partial^2 w}{\partial \beta^2} \tag{4.17}$$

$$M^* = (k_1D_{11} + k_2D_{12}) \frac{\partial u}{\partial \alpha} + (k_2D_{22} + k_1D_{12}) \frac{\partial v}{\partial \beta} + \tag{4.18}$$
$$+ (k_1^2D_{11} + 2k_1k_2D_{12} + k_2^2D_{22}) w - \frac{h^2}{120}\left(\frac{\partial \varphi}{\partial \alpha} + \frac{\partial \psi}{\partial \beta}\right)$$

The equations of equilibrium of an element of the shell are of the form

$$\frac{\partial T_1}{\partial \alpha} + \frac{\partial S}{\partial \beta} = 0, \qquad \frac{\partial T_2}{\partial \beta} + \frac{\partial S}{\partial \alpha} = 0, \qquad \frac{\partial M_1}{\partial \alpha} + \frac{\partial H}{\partial \beta} = N_1$$
$$\frac{\partial M_2}{\partial \beta} + \frac{\partial H}{\partial \alpha} = N_2, \qquad (k_1T_1 + k_2T_2) - \frac{\partial N_1}{\partial \alpha} - \frac{\partial N_2}{\partial \beta} = Z \tag{4.19}$$

Substituting expressions (4.11) to (4.16) into Equations (4.19), we obtain a system of differential equations of very shallow shells:

$$L_{11}(C_{ik})\, u + L_{12}(C_{ik})\, v + (k_1C_{11} + k_2C_{12}) \frac{\partial w}{\partial \alpha} - \tag{4.20}$$
$$- A_1\left[(k_1D_{11} + k_2D_{12}) \frac{\partial^3 w}{\partial \alpha^3} + (k_2D_{22} + k_1D_{12}) \frac{\partial^3 w}{\partial \alpha \partial \beta^2}\right] = A_1 \frac{h}{2} \frac{\partial Z}{\partial \alpha}$$

$$L_{22}(C_{ik})\, v + L_{12}(C_{ik})\, u + (k_2C_{22} + k_1C_{12}) \frac{\partial w}{\partial \beta} - \tag{4.21}$$
$$- A_2\left[(k_2D_{22} + k_1D_{12}) \frac{\partial^3 w}{\partial \beta^3} + (k_1D_{11} + k_2D_{12}) \frac{\partial^3 w}{\partial \beta \partial \alpha^2}\right] = A_2 \frac{h}{2} \frac{\partial Z}{\partial \beta}$$

$$(k_1C_{11} + k_2C_{12}) \frac{\partial u}{\partial \alpha} + (k_2C_{22} + k_1C_{12}) \frac{\partial v}{\partial \beta} + (k_1^2C_{11} + 2k_1k_2C_{12} + k_2^2C_{22})\, w -$$
$$- \frac{h^3}{12}\left(\frac{\partial \varphi}{\partial \alpha} + \frac{\partial \psi}{\partial \beta}\right) - (k_1A_1 + k_2A_2)\left[(k_1D_{11} + k_2D_{12}) \frac{\partial^2 w}{\partial \alpha^2} + \right. \tag{4.22}$$
$$\left. + (k_2D_{22} + k_1D_{12}) \frac{\partial^2 w}{\partial \beta^2}\right] = Z\left[1 + \frac{h}{2}(k_1A_1 + k_2A_2)\right]$$

$$L_{13}(D_{ik})\, w - \frac{h^2}{10}\left[a_{55}L_{11}(D_{ik})\, \varphi + a_{44}L_{12}(D_{ik})\, \psi\right] + \frac{h^3}{12} \varphi +$$
$$+ A_1\left[(k_1D_{11} + k_2D_{12}) \frac{\partial^2 u}{\partial \alpha^2} + (k_2D_{22} + k_1D_{12}) \frac{\partial^2 v}{\partial \alpha \partial \beta} + \right. \tag{4.23}$$
$$\left. + (k_1^2D_{11} + 2k_1k_2D_{12} + k_2^2D_{22}) \frac{\partial w}{\partial \alpha} - \frac{h^5}{120}\left(\frac{\partial^2 \varphi}{\partial \alpha^2} + \frac{\partial^2 \psi}{\partial \alpha \partial \beta}\right)\right] = 0$$

$$L_{23}(D_{ik})\, w - \frac{h^2}{10}\left[a_{44}L_{22}(D_{ik})\, \psi + a_{55}L_{12}(D_{ik})\, \varphi\right] + \frac{h^3}{12} \psi +$$
$$+ A_2\left[(k_2D_{22} + k_1D_{12}) \frac{\partial^2 v}{\partial \beta^2} + (k_2D_{11} + k_1D_{12}) \frac{\partial^2 u}{\partial \alpha \partial \beta} + \right. \tag{4.24}$$
$$\left. + (k_1^2D_{11} + 2k_1k_2D_{12} + k_2^2D_{22}) \frac{\partial w}{\partial \beta} - \frac{h^5}{120}\left(\frac{\partial^2 \psi}{\partial \beta^2} + \frac{\partial^2 \varphi}{\partial \alpha \partial \beta}\right)\right] = 0$$

Equations (4.20) to (4.24) constitute a complete system of five differential equations with respect to five unknown functions $u$, $v$, $w$, $\phi$, $\psi$.

The basic equations of the theory of very shallow shells may be represented also in the present formulation in the form of the mixed method

[2, 5, 6].

The equation of compatibility of strain of the middle surface of the shell is of the form

$$k_2\varkappa_1 + k_1\varkappa_2 + \frac{\partial^2\varepsilon_2}{\partial\alpha^2} - \frac{\partial^2\omega}{\partial\alpha\partial\beta} + \frac{\partial^2\varepsilon_1}{\partial\beta^2} = 0 \tag{4.25}$$

Examining Formulas (4.11) to (4.15), it is easy to see that the factor of the stiffness $C_{11}$ is the relative elongation of the middle surface $\epsilon_1$ of $C_{22}$ is the relative elongation $\epsilon_2$, and of $C_{66}$ is the shear of the middle surface $\omega$; further, the factor of the stiffness $D_{11}$ is the parameter characterizing the change of curvature of the middle surface $\kappa_1$, and finally, $D_{22}$ is the change of curvature $\kappa_2$.

Taking this into account, Equation (4.25) may be represented in the form

$$\frac{C_{22}}{\Omega}\frac{\partial^2T_1}{\partial\beta^2} - \frac{C_{12}}{\Omega}\frac{\partial^2T_1}{\partial\alpha^2} + \frac{C_{11}}{\Omega}\frac{\partial^2T_2}{\partial\alpha^2} - \frac{C_{12}}{\Omega}\frac{\partial^2T_2}{\partial\beta^2} - \frac{1}{C_{66}}\frac{\partial^2S}{\partial\alpha\partial\beta} - k_2\frac{\partial^2w}{\partial\alpha^2} -$$
$$- k_1\frac{\partial^2w}{\partial\beta^2} + \left(A_2\frac{C_{11}}{\Omega} - A_1\frac{C_{12}}{\Omega}\right)\frac{\partial^2T^*}{\partial\alpha^2} + \left(A_1\frac{C_{22}}{\Omega} - A_2\frac{C_{12}}{\Omega}\right)\frac{\partial^2T^*}{\partial\beta^2} = 0 \tag{4.26}$$
$$(\Omega = C_{11}C_{22} - C_{12}^2)$$

Let us introduce the stress function $F = F(\alpha, \beta)$, such that

$$T_1 = \frac{\partial^2F}{\partial\beta^2}, \quad T_2 = \frac{\partial^2F}{\partial\alpha^2}, \quad S = -\frac{\partial^2F}{\partial\alpha\partial\beta} \tag{4.27}$$

Then the first two equations of equilibrium (4.19) are satisfied identically and from the remaining three equations (4.19) and the equation of compatibility (4.26), by virtue of (4.14)-(4.16) and (4.27), we obtain a complete set of differential equations:

$$\nabla_rF - \frac{h^3}{12}\left(\frac{\partial\varphi}{\partial\alpha} + \frac{\partial\psi}{\partial\beta}\right) = Z \tag{4.28}$$

$$L_{13}(D_{ik})\,w - \frac{h^2}{10}\left[a_{55}L_{11}(D_{ik})\,\varphi + a_{44}L_{12}(D_{ik})\,\psi\right] + \frac{h^3}{12}\varphi +$$
$$+ A_1\Big[(k_1D_{11} + k_2D_{12})\frac{\partial^2u}{\partial\alpha^2} + (k_2D_{22} + k_1D_{12})\frac{\partial^2v}{\partial\alpha\,\partial\beta} +$$
$$+ (k_1^2D_{11} + 2k_1k_2D_{12} + k_2^2D_{22})\frac{\partial w}{\partial\alpha} - \frac{h^5}{120}\left(\frac{\partial^2\varphi}{\partial\alpha^2} + \frac{\partial^2\psi}{\partial\alpha\,\partial\beta}\right)\Big] = 0 \tag{4.29}$$

$$L_{23}(D_{ik})\,w - \frac{h^2}{10}\left[a_{44}L_{22}(D_{ik})\,\psi + \alpha_{55}L_{12}(D_{ik})\,\varphi\right] + \frac{h^3}{12}\psi +$$
$$+ A_2\Big[(k_2D_{22} + k_1D_{12})\frac{\partial^2v}{\partial\beta^2} + (k_1D_{11} + k_2D_{12})\frac{\partial^2u}{\partial\alpha\,\partial\beta} +$$
$$+ (k_1^2D_{11} + 2k_1k_2D_{12} + k_2^2D_{22})\frac{\partial w}{\partial\beta} - \frac{h^5}{120}\left(\frac{\partial^2\psi}{\partial\beta^2} + \frac{\partial^2\varphi}{\partial\alpha\,\partial\beta}\right)\Big] = 0 \tag{4.30}$$

$$L_2(C_{ik})F - \nabla_r w + \left(A_2\frac{C_{11}}{\Omega} - A_1\frac{C_{12}}{\Omega}\right)(k_1D_{11} + k_2D_{12})\frac{\partial^4 w}{\partial\alpha^4} +$$
$$+\left[\left(A_2\frac{C_{11}}{\Omega} - A_1\frac{C_{12}}{\Omega}\right)(k_2D_{22} + k_1D_{12}) + \left(A_1\frac{C_{22}}{\Omega} - A_2\frac{C_{12}}{\Omega}\right)(k_1D_{11} +\right.$$
$$\left.+ k_2D_{12})\right]\frac{\partial^4 w}{\partial\alpha^2\partial\beta^2} + \left(A_1\frac{C_{22}}{\Omega} - A_2\frac{C_{12}}{\Omega}\right)(k_2D_{22} + k_1D_{12})\frac{\partial^4 w}{\partial\beta^4} =$$
$$= -\frac{h}{2}\left[\left(A_2\frac{C_{11}}{\Omega} - A_1\frac{C_{12}}{\Omega}\right)\frac{\partial^2 Z}{\partial\alpha^2} + \left(A_1\frac{C_{22}}{\Omega} - A_2\frac{C_{12}}{\Omega}\right)\frac{\partial^2 Z}{\partial\beta^2}\right] \qquad (4.31)$$

where

$$\nabla_r = k_2\frac{\partial^2}{\partial\alpha^2} + k_1\frac{\partial^2}{\partial\beta^2}$$
$$L_2(C_{ik}) = \frac{C_{11}}{\Omega}\frac{\partial^4}{\partial\alpha^4} + \frac{C_{22}}{\Omega}\frac{\partial^4}{\partial\beta^4} + \left(\frac{1}{C_{66}} - 2\frac{C_{12}}{\Omega}\right)\frac{\partial^4}{\partial\alpha^2\partial\beta^2} \qquad (4.32)$$

Thus, the problem of a very shallow shell in the form of the mixed method has been reduced to a system of four differential equations with respect to four unknown functions $F$, $w$, $\phi$, $\psi$.

The pertinent values of the problem may be determined with the aid of Formulas (4.6)-(4.16). The tangential displacements $(u, v)$ which enter into the indicated formulas may be found from the equations

$$\frac{\partial u}{\partial\alpha} = \frac{C_{22}}{\Omega}\frac{\partial^2 F}{\partial\beta^2} - \frac{C_{12}}{\Omega}\frac{\partial^2 F}{\partial\alpha^2} - k_1 w + \left(A_1\frac{C_{22}}{\Omega} - A_2\frac{C_{12}}{\Omega}\right)\times \qquad (4.33)$$
$$\times\left[\frac{h}{2}Z + (k_1D_{11} + k_2D_{12})\frac{\partial^2 w}{\partial\alpha^2} + (k_2D_{22} + k_1D_{12})\frac{\partial^2 w}{\partial\beta^2}\right]$$

$$\frac{\partial v}{\partial\beta} = \frac{C_{11}}{\Omega}\frac{\partial^2 F}{\partial\alpha^2} - \frac{C_{12}}{\Omega}\frac{\partial^2 F}{\partial\beta^2} - k_2 w + \left(A_2\frac{C_{11}}{\Omega} - A_1\frac{C_{12}}{\Omega}\right)\times \qquad (4.34)$$
$$\times\left[\frac{h}{2}Z + (k_2D_{22} + k_1D_{12})\frac{\partial^2 w}{\partial\beta^2} + (k_1D_{11} + k_2D_{12})\frac{\partial^2 w}{\partial\alpha^2}\right]$$

5. Let us consider two examples as an illustration. Without loss of generality in the deductions and in the calculating procedure we consider examples of a transversely isotropic plate and shell, assuming that at each point of the plate (or shell) the plane of isotropy is parallel to the middle plane (or surface) of the plate (or shell). The following relations are valid for the elastic constants of the material of the plate (or shell) [1]:

$$B_{11} = B_{22} = B_{12} + 2B_{66} = \frac{E}{1-\nu^2} = E^\circ, \qquad B_{12} = \nu E^\circ$$
$$B_{66} = \frac{1-\nu}{2}E^\circ, \qquad B_{12} + B_{66} = \frac{1+\nu}{2}E^\circ \qquad (5.1)$$
$$B_{55} = B_{44} = G', \qquad A_1 = A_2 = -\frac{E^\circ}{E'}\nu'(1+\nu)$$

where $E$ is the modulus of elasticity in directions in the plane of isotropy, $E'$ is the modulus of elasticity in directions perpendicular to the middle surface, $\nu$ is Poisson's ratio in the plane of isotropy, $\nu'$ is Poisson's ratio characterizing the shortening in the plane of isotropy accompanying extension in the $\gamma$ direction, $G'$ is the shear modulus characterizing the distortion of angles between directions in the plane of isotropy and the direction $\gamma$.

*Example 1.* Let a rectangular plate ($a \times b$) be simply supported along the total contour, and be subjected to a loading which is distributed on the surface of the plate in accordance with the law

$$Z^+ = Z = q \sin\frac{\pi\alpha}{a} \sin\frac{\pi\beta}{b}, \qquad Z^- = 0 \tag{5.2}$$

where $q$ is the intensity of loading at the center of the plate ($\alpha = 1/2\ a$, $\beta = 1/2\ b$).

Assuming [3]

$$\varphi = B \cos\frac{\pi\alpha}{a} \sin\frac{\pi\beta}{b}, \qquad \psi = C \sin\frac{\pi\alpha}{a} \cos\frac{\pi\beta}{b}, \qquad w = A \sin\frac{\pi\alpha}{a} \sin\frac{\pi\beta}{b} \tag{5.3}$$

we satisfy the conditions of simple support and from the system of equations (3.3)-(3.5) and by virtue of (5.2), (5.3) for the deflection of the center of the plate we obtain

$$w = w_0 \left\{1 + \frac{\pi^2}{10} h^2 \left(\frac{1}{a^2} + \frac{1}{b^2}\right) \left[\frac{E^\circ}{G'} - \nu'(1+\nu)\frac{E^\circ}{E'}\right]\right\} \tag{5.4}$$

where

$$w_0 = \frac{12a^4b^4q}{\pi^4h^3E^\circ(a^2+b^2)^2}$$

is the deflection of the center of the plate determined with the aid of the classical theory of plates, that is, with the aid of a theory which is based on the hypothesis of the undeformable normals.

Examining Formula (5.4) it is easy to recognize that for certain values of the ratios $E^\circ/G'$, $E^\circ/E'$, $h/a$ the normal displacements, calculated on the basis of the classical theory of plates, may differ considerably from the corresponding displacements, calculated on the basis of the theory advanced here.

Indeed, the classical theory of anisotropic plates, containing an error of the order $h^2/a^2$ as compared to unity, as could be expected, is completely indifferent to ratios of the type $B_{ik}/B_{55}$, $B_{ik}/B_{44}$, $B_{ik}/B_{33}$ which appear in more rigorous theories of anisotropic plates with a numerical coefficient and the factor $h^2/a^2$ and may have considerably larger numerical values.

The theory advanced here for certain boundary conditions and for certain loadings may also be used for the analysis of thick plates. For example, in the case of a thick isotropic square plate ($h/a = 1/3$, $\nu = 0.3$, $a = b$), when the plate is subjected to a loading which on the surface of the plate ($\gamma = 1/2\ h$) is distributed in accordance with the law (5.2), we have the following values for the deflection of the center of the plate:

| | Exact theory [7] | Proposed theory | Theory [8] | Theory [3] | Classical theory |
|---|---|---|---|---|---|
| $W(E/qh) =$ | 3.49 | 3.50 | 3.56 | 3.69 | 2.27 |

It is seen that even for such a thick plate ($h/a = 1/3$) the theory advanced here yields an insignificant error (of the order 0.3%). The error of the theory of isotropic plates of medium thickness [8] reaches 2%, and of the approximate theory [3], which does not take into account the influence of normal stress $\sigma_\gamma$, reaches 6%. The error of the classical theory, however, is equal to 35%.

The calculations indicate that the theory presented also yields good results in the analysis of thick plates. However, it may not be considered a theory of thick plates or plates of medium thickness without qualification; for indicated plates, difficulties will arise in connection with boundary conditions [3,9].

Finally, let us indicate that the correction introduced into the classical theory, due to the effect of transverse shear, is more significant than the correction due to the effect of normal stress $\sigma_\gamma$. For example, in the problem of the thick plate considered above, the correction due to the effect of $\sigma_\gamma$ is of the order of 5%, while the correction due to the effect of transverse shear reaches 30%. Numerous calculations, carried out for actual anisotropic plates, confirm the discussions expressed above. In this connection we assume that in the analysis of thin anisotropic plates (and shells) all those somewhat illogical theories are applicable in which phenomena associated with the stress $\sigma_\gamma$ are not taken into account on purpose.

*Example* 2. Let a very shallow, transversely isotropic, rectangular (in plan form) shell be simply supported along its contour and let it be subjected to a normally applied loading which is distributed in accordance to the law (5.2) on the surface of the shell ($\gamma = 1/2\ h$). In this example we shall neglect the influence of the stress $\sigma_\gamma$. Thus, it will be sufficient to assume in all equations and formulas $A_1 = A_2 = 0$.

Letting [4]

$$\varphi = B\cos\frac{\pi\alpha}{a}\sin\frac{\pi\beta}{b}, \qquad \psi = C\sin\frac{\pi\alpha}{a}\cos\frac{\pi\beta}{b}$$

$$u = M\cos\frac{\pi\alpha}{a}\sin\frac{\pi\beta}{b}, \qquad v = N\sin\frac{\pi\alpha}{a}\cos\frac{\pi\beta}{b} \tag{5.5}$$

$$w = A\sin\frac{\pi\alpha}{a}\sin\frac{\pi\beta}{b}$$

the conditions of free support will be satisfied and from the system of equations (4.20)-(4.24) we obtain, by virtue of (5.2) and (5.5), for the normal displacement of the center of the shell ($\alpha = a/2$, $\beta = b/2$)

$$w = w_0[1 + h^*] \tag{5.6}$$

where

$$h^* = \frac{\frac{h^4}{120}\frac{E^\circ}{G'}\left(\frac{\pi^2}{a^2} + \frac{\pi^2}{b^2}\right)^5}{\frac{h^2}{12}\left(\frac{\pi^2}{a^2} + \frac{\pi^2}{b^2}\right)^4 + (1-\nu^2)\left(k_2\frac{\pi^2}{a^2} + k_1\frac{\pi^2}{b^2}\right)^2\left[1 + \frac{h^2}{10}\frac{\overline{E}^\circ}{G'}\left(\frac{\pi^2}{a^2} + \frac{\pi^2}{b^2}\right)\right]} \tag{5.7}$$

$w_0$ is the normal displacement of the center of the shell, determined with the aid of the theory which is based on the hypothesis of undeformable normals.

Examining Formulas (5.6) and (5.7), we note that with an increase in the rise of the shell (that is, with an increase of the ratios $a/R_1$, $a/R_2$, the error committed in adopting the hypothesis of undeformable normals is decreased. This error reaches its maximum value in the case of a plate ($k_1 = 1/R_1 = 0$, $k_2 = 1/R_2 = 0$). Here the reason for this is due to the fact that as the rise of the shell increases the influence of flexural parameters upon the state of stress of the shell decreases, which means a decrease in the influence of shear forces $N_1$ and $N_2$, that is, of shear stresses $\tau_{\alpha\gamma}$ and $\tau_{\beta\gamma}$ which produce the influence of transverse shear. Generally speaking, the smaller the influence of flexural phenomena upon the state of stress of the shell the smaller the "correction" of the classical theory of shells due to phenomena of transverse shear.

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*Translated by* G.H.